Thick self-gravitating plane-symmetric domain walls

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(February 7, 2008)

Abstract

We investigate a self-gravitating thick domain wall for a $\lambda\Phi^4$ potential. The system of scalar and Einstein equations admits two types of non-trivial solutions: domain wall solutions and false vacuum-de Sitter solutions. The existence and stability of these solutions depends on the strength of the gravitational interaction of the scalar field, which is characterized by the number ϵ. For $\epsilon \ll 1$, we find a domain wall solution by expanding the equations of motion around the flat spacetime kink. For "large" ϵ, we show analytically and numerically that only the de Sitter solution exists, and that there is a phase transition at some ϵ_{max} which separates the two kinds of solution. Finally, we comment on the existence of this phase transition and its relation to the topology of the domain wall spacetime.

INTRODUCTION

The spacetime of cosmological domain walls has now been a subject of interest for more than a decade since the work of Vilenkin and Ipser and Sikivie [1,2], who used Israel's thin wall formalism [3] to compute the gravitational field of an infinitesimally thin planar domain wall. This revealed the gravitating domain wall as a rather interesting object: although the scalar field adopts a static solitonic form, the spacetime cannot be static if one imposes a reflection symmetry around the defect [1,2], but displays a de Sitter expansion in any plane parallel to the wall. Moreover, there is a cosmological event horizon at finite proper distance from the wall; this horizon provides a length scale to the coupled system of the Einstein and scalar field equations for a thin wall.

After the original work by Vilenkin, Ipser and Sikivie [1,2] for thin walls, attempts focussed on trying to find a perturbative expansion in the wall thickness [4,5]. With the proposition by Hill, Schramm and Fry [6] of a late phase transition with thick domain walls, there was some effort in finding exact thick solutions [7,8]; however, these walls were

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supposed to be thick by virtue of the low temperature of the phase transition, which means that the scalar couples very weakly to gravitation. The suggestion that the cores of defects created near the Planck time could undergo an inflationary expansion [9,10] then reopened the question of thick domain walls (where now thick means relative to the wall's natural de Sitter horizon). This time, the high temperature of the phase transition ensures that the scalar field in this case interacts very strongly with gravity. Here, we consider gravitating thick domain wall solutions with planar and reflection symmetry in the Goldstone model; a more detailed discussion can be found in our paper [11], where we consider general potentials and de Sitter/anti-de Sitter background spacetimes.

DOMAIN WALLS IN FLAT SPACETIME

In order to fix the notation let us first briefly review a domain wall in flat space-time [12]. Consider a flat metric η_{ab} with signature $(+, -, -, -)$ and spacetime coordinates $x^a = \{t, x, y, z\}$. The matter Lagrangian will be given by the $\lambda \Phi^4$ Lagrangian

$$\mathcal{L} = \eta^{ab} \nabla_a \Phi \nabla_b \Phi - V(\Phi) \tag{1}$$

$$V(\Phi) = \lambda \left(\Phi^2 - \eta^2 \right)^2, \tag{2}$$

where $\Phi = \Phi(z)$ is a real scalar field. For a potential with a non-trivial degenerate set of minima, such as (2), one gets domain wall solutions because the vacuum manifold $\mathcal{M} = \{\pm\eta\}$ is not connected. We scale out the dimensionful symmetry breaking scale η by letting $X = \Phi/\eta$ and we set the wall's width $w = \frac{1}{\sqrt{\lambda}\eta}$ to unity (thus measuring distances in wall units rather than Planck units). Then the matter langrangian takes the simplified form

$$\mathcal{L} = -(X')^2 - (X^2 - 1)^2 \tag{3}$$

with equation of motion

$$X'' - 2X(X^2 - 1) = 0. \tag{4}$$

The well known kink solution is given by $X(z) = \tanh z$, an odd function approaching exponentially the true vacua ± 1 as $z \to \pm\infty$ (see figure 1).

GRAVITATION AND GENERAL SETTING OF THE PROBLEM

Now let us couple our flat space-time domain wall solution to gravity in a minimal way. Consider a domain wall with local planar symmetry, reflection symmetry around the wall's core at $z = 0$. The matter langrangian is given by,

$$\mathcal{L} = g^{ab} \nabla_a X \nabla_b X - (X^2 - 1)^2 \tag{5}$$

and coupled to gravity via a metric admitting these symmetries, of components g_{ab}. We suppose again that $X = X(z)$, a static field and the metric g_{ab} is given by,

$$ds^2 = A^2(z) \, dt^2 - B^2(z, t) \left(dx^2 + dy^2 \right) - dz^2 \tag{6}$$
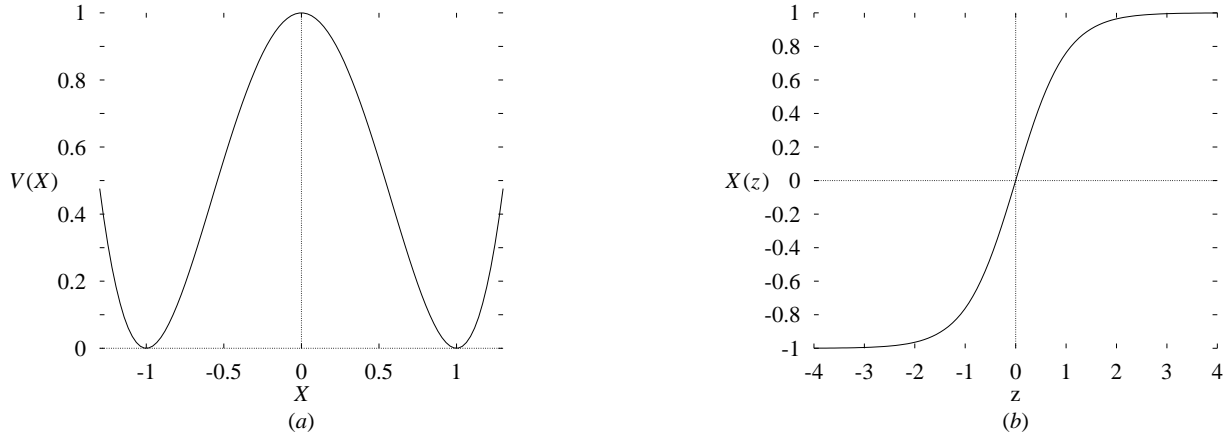
Figure 1. (a) The potential $V(X) = (X^2 - 1)^2$. (b) The flat spacetime kink solution, $X(z) = \tanh(z)$.

with z measuring the proper distance from the wall.

Now in order to find a domain wall solution we have to solve the coupled system of differential equations consisting of the Einstein and scalar field equations namely,

$$R_{ab} = \epsilon \left[2X_{,a}X_{,b} - g_{ab}(X^2 - 1)^2 \right] \tag{7a}$$

$$\Box X + 2X(X^2 - 1) = 0, \tag{7b}$$

where $\epsilon = 8\pi G\eta^2$ is a dimensionless parameter which we call gravitational strength parameter and which characterises the gravitational interaction of the Higgs field. Note that the Ricci tensor R_{ab} is generated by wall matter via the Einstein equations, which is the essence of self-gravity.

Since the field profile function X is time independent, one sees from the Einstein equations that the metric reduces to,

$$ds^2 = A^2(z)\, dt^2 - A^2(z)e^{2kt}\left(dx^2 + dy^2\right) - dz^2. \tag{8}$$

Then (7) reduces to,

$$\frac{A''}{A} = -\frac{\epsilon}{3}\left[2X'^2 + (X^2 - 1)^2\right] \tag{9a}$$

$$X'' + 3\frac{A'}{A}X' = 2X(X^2 - 1) \tag{9b}$$

$$\left(\frac{A'}{A}\right)^2 = \frac{k^2}{A^2} + \epsilon\left[X'^2 - (X^2 - 1)^2\right] \tag{9c}$$

where the third equation is a constraint equation giving the value of the constant k. One can see that (9) admits for all ϵ the false vacuum de Sitter solution, $X = 0$, $A(z) = \cos kz$, $k^2 = \epsilon/3$; here, the role of the cosmological constant is played essentially by the parameter ϵ. For a domain wall solution we demand the following boundary conditions,

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- $X(z)$ is an odd function (for a topological solution);
- $|X| \longrightarrow |X_0| < 1$, as $z \longrightarrow \pm z_h$;
- $X'(z_{\mathrm{h}}) = 0$;
- $A(0) = 1$, $\qquad A'(0) = 0$.

We are expecting that in the presence of gravity our coordinate system will break down at some z_{h}, representing the proper distance of the wall's core to the event horizon. This means that the scalar field will not fall all the way down the potential to its minimum value, ± 1, within the range of validity of our coordinates. Moreover in order for the solution to be non-singular we will have to suppose that $X'(z_{\mathrm{h}}) = 0$. The conditions on $A(z)$ result from reflection symmetry and are always valid from the regularity of the metric.

PERTURBATIVE SOLUTION FOR WEAK GRAVITY

When $\epsilon \ll 1$, corresponding to weak self-gravity, we can expand the unknown fields in powers of ϵ,

$$X = X_0 + \epsilon\, X_1 + O\left(\epsilon^2\right) \tag{10}$$

$$A = A_0 + \epsilon\, A_1 + O\left(\epsilon^2\right), \tag{11}$$

where to zeroth order we have of course the flat space-time solution,

$$X_0 = \tanh(z), \qquad A_0 = 1. \tag{12}$$

The field equations (9) to first order in ϵ give

$$A_1'' = -\frac{1}{3}\left[2X_0'^2 + (X_0^2 - 1)^2\right] \tag{13}$$

$$X_1'' = -3A_1'X_0' + 2X_1(3X_0^2 - 1) \tag{14}$$

$$k^2 = \epsilon^2\left[A_1'^2 + \frac{2}{3}(X_1 X_0'' - X_0' X_1')\right] \tag{15}$$

Solving in turn these equations we obtain,

$$X(z) = \tanh z - \frac{\epsilon}{2}\operatorname{sech}^2 z\left[z + \frac{1}{3}\tanh z\right] + \mathrm{O}\left(\epsilon^2\right) \tag{16a}$$

$$A(z) = 1 - \frac{\epsilon}{3}\left[\ln\cosh z - \frac{1}{2}\tanh^2 z\right] + \mathrm{O}\left(\epsilon^2\right) \tag{16b}$$

$$k = \pm\frac{2}{3}\epsilon + \mathrm{O}\left(\epsilon^2\right). \tag{16c}$$

Then since A_1'' is negative, A asymptotes $1 - kz$ for large z. Thus at proper distance from the wall given by, $z_{\mathrm{h}} = \frac{1}{k} \sim \epsilon^{-1}$ we have an event horizon since $A(z) \underset{z \to 1/k}{\longrightarrow} 0$ and $g_{tt} = A^2$. Figure 2 compares the perturbative solution (16) with the numerical solution.
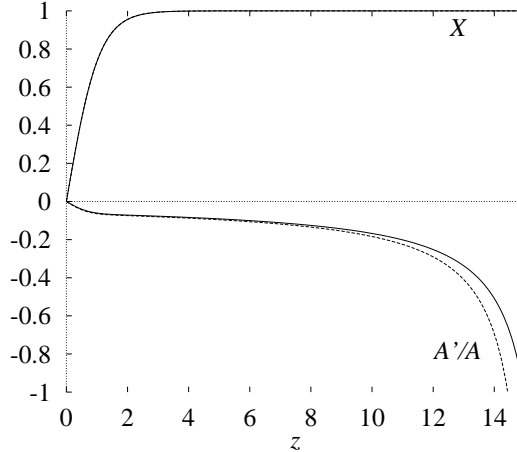
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Figure 2. Comparison between the numerical solution (solid lines) and the series solution (16). At this scale the two solutions for X are superposed.

STRONGLY SELF-GRAVITATING WALLS

Let us now consider the case of strong self-gravity, corresponding to $\epsilon = \mathrm{O}(1)$, for supermassive walls forming near the Planck scale. For such large ϵ where $\eta \sim M_{\mathrm{Pl}}$ our perturbative analysis breaks down. Nevertheless generically one can consider that since for smaller values of ϵ, $z_{\mathrm{h}} \sim \epsilon^{-1}$,

$$\text{bigger } \epsilon \qquad \longleftrightarrow \qquad \text{smaller } z_{\mathrm{h}}.$$

For ϵ varying at this range we can only solve (9) numerically, an example for $\epsilon = 0.9$ is shown in figure 3.

For large enough ϵ the event horizon distance becomes of the same order as the wall's thickness, $z_{\mathrm{h}} = \mathrm{O}(w)$. In (9a, 9c), the effect of the matter energy momentum tensor is intensified, which in turn increases the effect of geometry, hence the variation of A. Then, in (9b), for a non-singular solution X to exist, X' has to tend to zero faster than A as we approach the horizon. We have then two distinct possibilities: firstly, the scalar field ignores the geometry and fluctuates around the false vacuum with an odd parity; the metric in the wall's core is then approximately de Sitter. Secondly, there is a phase transition in the behaviour of X; that is for some ϵ_{max} the wall solution ceases to exist and the only non-singular solution turns out to be exactly the de Sitter one. To put it in a nutshell, either the field X rolls significantly from the false vacuum or not at all. Note that when investigating the case of a non-gravitating domain wall in a de Sitter background, Basu and Vilenkin [13] have precisely observed such a phase transition. The issue of whether a domain wall can survive in a Friedmann-Robertson-Walker (FRW) universe has also been analysed in the case of Euclidean instantons on a de Sitter background including self-gravity [14,15].

Let us now prove that—given the symmetries that we impose to our solutions—it is the second possibility that holds. To do this, we first prove that in some range of the parameter ϵ the de Sitter solution is unstable to decay into a wall solution; then, we find an estimate of the value ϵ_{max} at which domain wall solutions cease to exist.
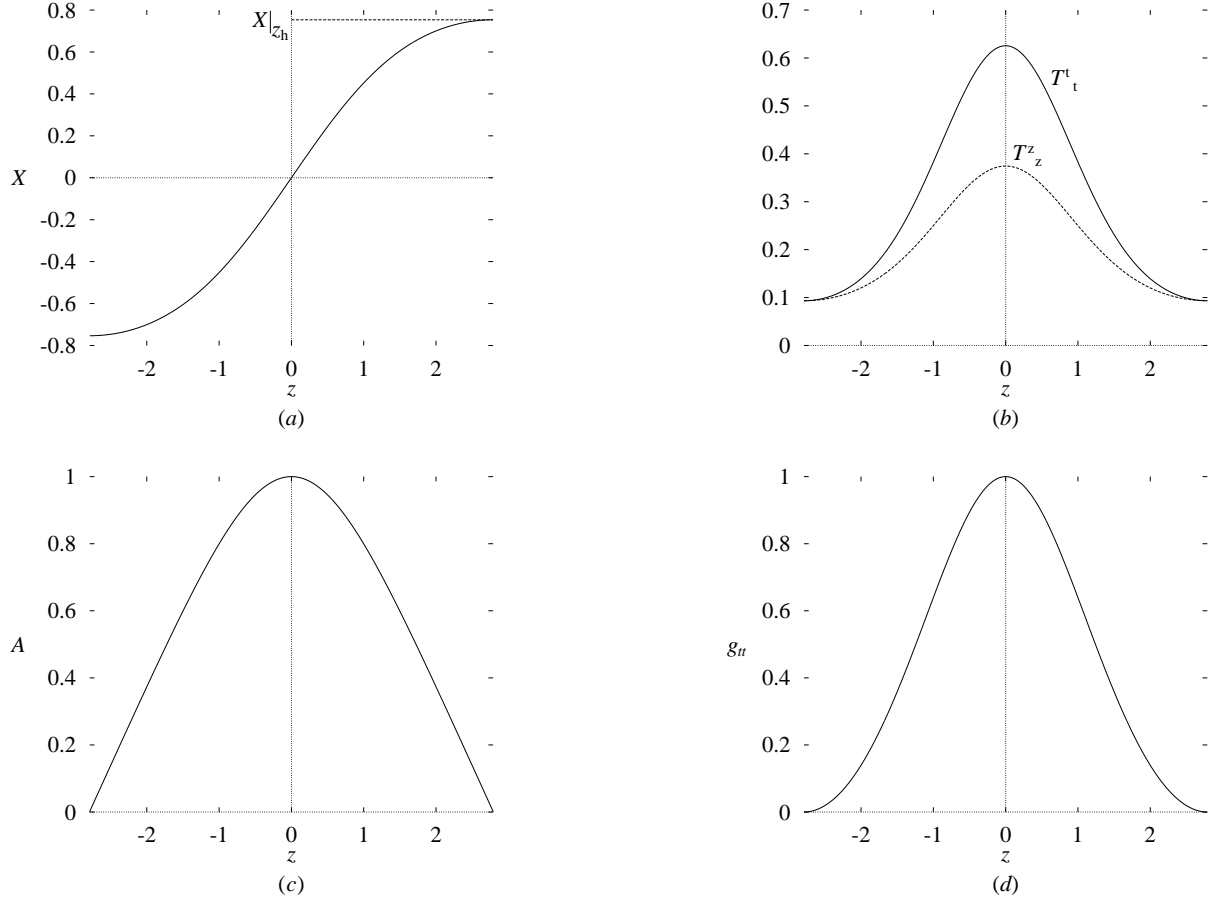
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Figure 3. Solution of the coupled scalar–Einstein equations for $\epsilon = 0.9$. (a) The Higgs field; (b) The diagonal components of the energy-momentum tensor $T^0{}_0 = T^x{}_x = T^y{}_y$ and $T^z{}_z$; (c) The function $A(z)$; (d) The metric function $g_{tt}(z) = A^2(z)$.

Stability of the false vacuum-de Sitter solution

The de Sitter solution is given by

$$ds^2 = \cos^2(kz)\, dt^2 - e^{2kt} \cos^2(kz) \left(dx^2 + dy^2 \right) - dz^2 \tag{17}$$

$$X = 0 \tag{18}$$

$$k^2 = \epsilon/3. \tag{19}$$

The false vacuum $X = 0$ is an *unstable* solution to wall formation if there exists a perturbation of X, say $\xi(z,t)$, which is an odd function of z and which is increasing in an unbounded manner with time. Consider then $X = \Delta\xi(z,t)$, with Δ an infinitesimally small parameter. We note that Einstein equations appear in order $O(\Delta^2)$ and can be neglected; the linearised *time-dependent* field equation to order Δ with respect to ξ gives,

$$\xi'' - 3k\tan(kz)\xi' - \sec^2(kz)[\ddot{\xi} + 2k\dot{\xi}] + 2\xi = 0. \tag{20}$$

We have a solution given by

$$\xi = e^{k\nu t} \sin kz (\cos kz)^\nu, \tag{21}$$

where

$$\nu = -\frac{5}{2} + \frac{1}{2}\sqrt{9 + 8/k^2}. \tag{22}$$

Then $\xi(z,t)$ is an odd function in z and if $\nu > 0$ it is exponentially increasing in time, so that the de Sitter solution $X = 0$ is unstable to wall formation. Now $k^2 = \epsilon/3$, and we find that for $\epsilon < 3/2$, de Sitter solution is unstable to wall formation. So it is more energetically favorable for the field X to roll to a domain wall solution than to remain in the false vacuum de Sitter solution for $\epsilon < 3/2$. This is to be expected since the false vacuum-de Sitter solution is inherently unstable and we have found for small ϵ domain wall solutions. Let us now turn to the wall solution.

Existence of wall solutions

As we have already noted, a domain wall solution requires an odd scalar field X such that X is non-singular ($X'(z_{\mathrm{h}}) = 0$) and non-trivial ($X'(0) > 0$).

Taking the derivative of the scalar equation, we get

$$\begin{aligned}
X''' &= -3\frac{A'}{A}X'' + X'\left[-3\frac{A''}{A} + 3(\frac{A'}{A})^2 + 6X^2 - 2\right] \\
&= -3\frac{A'}{A}X'' + X'F(z)
\end{aligned} \tag{23}$$

where

$$F(z) = \left[3\epsilon X'^2 + 3\frac{k^2}{A^2} + 6X^2 - 2\right] \tag{24}$$

and we have used the Einstein equations to replace A''/A and A'/A in $F(z)$.

At $z = 0$ we have,

$$3k^2 = \epsilon[1 - X'(0)^2] \tag{25}$$

from the constraint (9c); hence

$$X'''(0) = X'(0)[3\epsilon - 6k^2 - 2] > X'(0)[\epsilon - 2]. \tag{26}$$

For $\epsilon > 2$, $X'''(0) > 0$, so $X'' > 0$ and X' is increasing away from $z = 0$. Moreover, $F(z)$ is strictly increasing away from $z = 0$ and thus X' can never be zero at the horizon.

We can refine numerically this estimate. We find that precisely at $\epsilon = 3/2$ there is a phase transition in the behavior of X (figure 4): when gravity is very strong the domain wall solution ceases to exist becoming singular and we are left with the false vacuum-de Sitter solution.
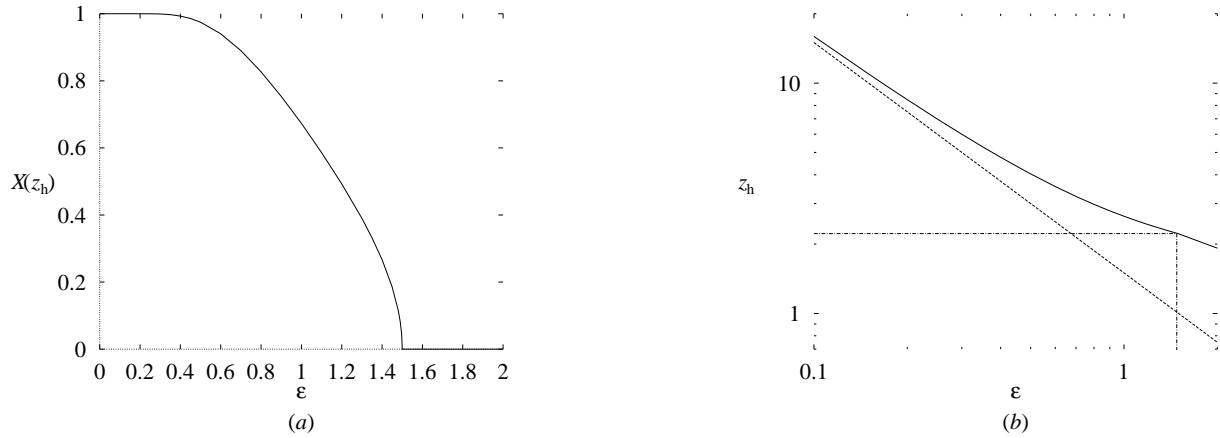

Figure 4. The phase transition of the Higgs field: (a) shows the value of the Higgs field at the wall's horizon, and (b) shows the proper distance to the horizon, z_{h}, both in function of ϵ.

CONCLUDING REMARKS

For a double well potential and $X = X(z)$, we have found:

- perturbative solutions for small ϵ,
- domain wall solutions for $\epsilon < 3/2$,
- that at $\epsilon = 3/2$, X undergoes a second order phase transition, and
- that for $\epsilon > 3/2$, domain wall solutions do not exist and the spacetime is de Sitter.

This phenomenon of phase transition in the behaviour of X is related to the topology of the gravitating wall (see also [16]) and the de Sitter spaces. Indeed the de Sitter spacetime can be pictured as a four-dimensional hyperboloid embedded in five-dimensional flat spacetime. Then the spatial sector of the metric is just the three-sphere which is compact.

It turns out [11] that the domain wall spacetime can also be viewed as a "squashed" hyperboloid in flat five-dimensional spacetime. Again, topologically, this is S^3 and the domain wall's space is compactified by gravity. Therefore, the $t = $ constant slices of spacetime can be pictured as a squashed ellipsoid with two characteristic lengths: the wall's width and the distance to the horizon, which varies with ϵ. As gravity increases, the proper distance to the horizon decreases and becomes eventually comparable to the wall's width. Then the phase transition occurs as the two length scales become identical, and space becomes exactly the de Sitter three-sphere.

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